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INTRODUCTION

Since the white noise calculus initiated by T. Hida [6] is based on Schwartz type distribution theory on Gaussian space (the foundation is due to I. Kubo and S. Takenaka [14]), it has been expected to provide a nice framework of calculus on Boson Fock space or an operator formalism of quantum physics.

In fact, one of the most significant features of white noise calculus lies in the possible use of pointwisely defined creation and annihilation operators. This has been improved to establish a general theory of operators on white noise functionals ([8], [9], [24], [25], [26] and [28] where the full description is given). During the study the key role has been played by an *integral kernel operator* with distribution κ as integral kernel:

$$(0\text{-}1)\qquad \Xi_{l,m}(\kappa)=\int_{T^{l+m}}\kappa(s_1,\cdots,s_l,t_1,\cdots,t_m)\partial_{s_1}^*\cdots\partial_{s_l}^*\partial_{t_1}\cdots\partial_{t_m}ds_1\cdots ds_l dt_1\cdots dt_m,$$

where ∂_s^* and ∂_t are creation and annihilation operators at points s and t, respectively. The use of distributions as integral kernels leads us to the theory of Fock expansion. Namely, in white noise calculus every continuous operator on white noise functionals admits an infinite series expansion in terms of integral kernel operators with precise estimate of the convergence. Moreover, having a nice characterization theorem for operator symbols, we can check easily whether or not an operator on Fock space defined only on the exponential vectors comes into our framework. (The theory is outlined in Sections 3–4.)

However, the discussion has been so far restricted to scalar-valued white noise functionals and little attention has been given to vector-valued ones. Meanwhile, we have obtained good motives to consider such generalization. First, calculus on Fock space has been considerably developed under the name of quantum probability and it has an interesting application to a quantum interacting system described on the tensor product of Boson Fock space and a so-called initial Hilbert space (in this connection see [11], [20], [21], [22], [30] and references cited therein; for a brief introduction to quantum probability theory see e.g., [33], [34]). Secondly in [1] A. Arai studied infinite dimensional Dirac operators defined on Boson-Fermion Fock space with application to supersymmetric quantum field theory, see also [2]. It is highly plausible that the idea of distributions on Gaussian space makes the discussion clearer. In fact, Z.-Y. Huang [10] discussed quantum Itô formula in terms of white noise calculus though the discussion is restricted to the case of scalar-valued functionals.

The main purpose of this paper is to propound a theory of vector-valued distribution theory on Gaussian space in line with white noise calculus. The first question is, obviously, to find a suitable vector space in which the distributions under consideration take values. In this paper we take a standard countably Hilbert space (see §1.2 for definition) for some special reasons. First of all, taking application into account, we must not exclude Hilbert spaces. Secondly, a standard countably Hilbert space possesses nice properties from the viewpoint of topological vector spaces, in particular, the theory of topological tensor products can be applied effectively. Finally, notation and results established for scalar-valued functionals help the study of vector-valued case very much.

The use of pointwisely defined creation and annihilation operators is not a new idea and there are closely related works, see **[4]** and references cited therein. In particular, Krée's work **[12]** is based on the theory of nuclear spaces and has much in common with ours, see also **[23]**. Nevertheless, an advantage of our theory is found in the theory of Fock expansion.

In Section 1 we assemble general notations and give the definition of a standard countably Hilbert space. Sections 2–4 are devoted to an overview of operator theory on scalar-valued white noise functionals. In Section 5 we discuss the vector-valued case, for the full description of the matters see the forthcoming paper **[29]**.

1. PRELIMINARIES

1.1. General notations. For a real vector space $\mathfrak{X}$ we denote its complexification by $\mathfrak{X}_{\mathbf{C}}$. Unless otherwise stated the dual space $\mathfrak{X}^*$ of a locally convex space $\mathfrak{X}$ is assumed to carry the strong dual topology. The canonical bilinear form on $\mathfrak{X}^* \times \mathfrak{X}$ is denoted by $\langle \cdot , \cdot \rangle$ or by similar symbols. When H is a complex Hilbert space, in order to avoid notational confusion we do not use the hermitian inner product but the $\mathbf{C}$-bilinear form on $H \times H$.

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be locally convex spaces. The π-topology is by definition the strongest locally convex topology on their algebraic tensor product $\mathfrak{X} \otimes_{\mathrm{alg}} \mathfrak{Y}$ such that the canonical bilinear map $\mathfrak{X} \times \mathfrak{Y} \to \mathfrak{X} \otimes_{\mathrm{alg}} \mathfrak{Y}$ is continuous. The completion of $\mathfrak{X} \otimes_{\mathrm{alg}} \mathfrak{Y}$ with respect to the π-topology is called *π-tensor product* and denoted by $\mathfrak{X} \otimes_{\pi} \mathfrak{Y}$. While, for two Hilbert spaces H and K we denote by $H \otimes K$ their Hilbert space tensor product. When there is no danger of confusion, $\mathfrak{X} \otimes_{\pi} \mathfrak{Y}$ is also denoted by $\mathfrak{X} \otimes \mathfrak{Y}$ for simplicity. For a locally convex space $\mathfrak{X}$ let $\mathfrak{X}^{\widehat{\otimes} n} \subset \mathfrak{X}^{\otimes n}$ the closed subspace of symmetric tensor products and let $(\mathfrak{X}^{\otimes n})^*_{\mathrm{sym}}$ be the space of continuous linear functionals on $\mathfrak{X}^{\otimes n}$ which are symmetric.

For two toplogical vector spaces $\mathfrak{X}$ and $\mathfrak{Y}$ let $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ stand for the space of continuous linear operators from $\mathfrak{X}$ into $\mathfrak{Y}$. The space $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ is equipped with the topology of uniform convergence on every bounded subset in $\mathfrak{X}$.

1.2. Standard countably Hilbert space. Let H be a (real or complex) Hilbert space with norm $|\cdot|_0$ and let A be a positive selfadjoint operator on H with $\inf \mathrm{Spec}\,(A) > 0$. Since $\mathrm{Spec}\,(A)$ is closed, the last condition is equivalent to that A admits a dense range and bounded inverse. According to the standard spectral theory we may define a selfadjoint operator A^p for any $p \in \mathbf{R}$ with maximal domain in H. Note that $\mathrm{Dom}\,(A^p) = H$ for $p < 0$. We then put

$$|\xi|_p = |A^p \xi|_0\,, \qquad \xi \in \mathrm{Dom}\,(A^p), \qquad p \in \mathbf{R}.$$

For $p \geq 0$ let E_p be the Hilbert space $\mathrm{Dom}\,(A^p)$ with the norm $|\cdot|_p$ and let E_{-p} be the completion of H with respect to $|\cdot|_{-p}$. These Hilbert spaces satisfy the natural inclusion relations:

$$\cdots \subset E_q \subset \cdots \subset E_p \subset \cdots \subset E_0 = H \subset \cdots \subset E_{-p} \subset \cdots \subset E_{-q} \subset \cdots\,, \qquad 0 \leq p \leq q.$$

Then, in an obvious manner,

$$E = \operatorname*{proj\,lim}_{p \to \infty} E_p = \bigcap_{p \geq 0} E_p$$

becomes a countably Hilbert space (abbr. CH-space). Since a general CH-space (see Gelfand-Vilenkin [5] for definition) is not necessarily of this type, we say that E is the *standard* CH-space constructed from a pair (H, A). It is known that E^* is isomorphic to the inductive limit:

$$E^* \cong \operatorname*{ind\,lim}_{p \to \infty} E_{-p} = \bigcup_{p \geq 0} E_{-p}.$$

A standard CH-space E constructed from (H, A) is nuclear if and only if A^{-r} is of Hilbert-Schmidt type for some $r > 0$. In that case we obtain a Gelfand triple $E \subset H \subset E^*$.

2. WHITE NOISE FUNCTIONALS

2.1. Gaussian space. Let T be a topological space with a Borel measure $\nu(dt) = dt$ and let $H = L^2(T, \nu; \mathbf{R})$ be the real Hilbert space of all ν-square integrable functions on T. The inner product is denoted by $\langle \cdot, \cdot \rangle$ and the norm by $|\cdot|_0$. We think of T as a time parameter space when it is an interval, or more generally as a field parameter space.

Let A be a positive selfadjoint operator on H with Hilbert-Schmidt inverse. Then there exist an increasing sequence of positive numbers $0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ and a complete orthonormal basis $\{e_j\}_{j=0}^{\infty}$ for H such that $Ae_j = \lambda_j e_j$ and

$$\delta \equiv \left(\sum_{j=0}^{\infty} \lambda_j^{-2} \right)^{1/2} = \left\| A^{-1} \right\|_{\mathrm{HS}} < \infty.$$

Let E be the standard CH-space constructed from (H, A). Since A^{-1} is of Hilbert-Schmidt type by assumption, E becomes a nuclear Fréchet space and we obtain a Gelfand triple $E \subset H = L^2(T, \nu; \mathbf{R}) \subset E^*$. The canonical bilinear form on $E^* \times E$ is also denoted by $\langle \cdot, \cdot \rangle$.

By construction each $\xi \in E$ is a function on T determined up to ν-null functions. This hiders from introducing a delta-function which is indispensable to our discussion. Accordingly we are led to the following hypotheses:

(H1) For each $\xi \in E$ there exists a unique continuous function $\tilde{\xi}$ on T such that $\xi(t) = \tilde{\xi}(t)$ for ν-a.e. $t \in T$.

Once this is satisfied, we always assume that every element in E is a continuous function on T and do not use the symbol $\tilde{\xi}$. We further need:

(H2) For each $t \in T$ a linear functional $\delta_t : \xi \mapsto \xi(t)$, $\xi \in E$, is continuous, i.e., $\delta_t \in E^*$;

(H3) The map $t \mapsto \delta_t \in E^*$, $t \in T$, is continuous.

(Recall that E^* is equipped with the strong dual topology.) Under (H1)-(H2) the convergence of a sequence in E implies the pointwise convergence as functions on T. If we have (H3) in addition, the convergence is uniform on every compact subset of T. Moreover, (H1)-(H3) are preserved under forming tensor products, see [**27**].

By another reason (see §2.3) we need one more assumption:

(S) $\lambda_0 = \inf \operatorname{Spec}(A) > 1$.

The constant number

$$0 < \rho \equiv \lambda_0^{-1} = \left\| A^{-1} \right\|_{\mathrm{OP}} < 1$$

will be often used together with the inequality $|\xi|_p \le \rho^q |\xi|_{p+q}$, $\xi \in E$, $p \in \mathbb{R}$, $q \ge 0$.

By the Bochner-Minlos theorem (e.g., [**7**]) there exists a unique probability measure μ on E^* (equipped with the Borel σ-field) such that

$$\exp\left(-\frac{1}{2}|\xi|_0^2\right) = \int_{E^*} e^{i\langle x, \xi\rangle} \mu(dx), \qquad \xi \in E.$$

This μ is called the *Gaussian measure* and the probability space (E^*, μ) is called the *Gaussian space*. We put $(L^2) = L^2(E^*, \mu; \mathbf{C})$ for simplicity.

2.2. Wiener-Itô decomposition. The canonical bilinear form on $(E^{\otimes n})^* \times (E^{\otimes n})$ is denoted by $\langle \cdot, \cdot \rangle$ again and its bilinear extension to $(E_{\mathbf{C}}^{\otimes n})^* \times (E_{\mathbf{C}}^{\otimes n})$ is also denoted by the same symbol.

For $x \in E^*$ let $: x^{\otimes n} :$ be defined as a unique element in $(E^{\otimes n})^*_{\mathrm{sym}}$ satisfying

$$\text{(2-1)} \qquad \phi_\xi(x) \equiv \sum_{n=0}^{\infty} \left\langle : x^{\otimes n} :, \frac{\xi^{\otimes n}}{n!} \right\rangle = \exp\left(\langle x, \xi \rangle - \frac{1}{2} \langle \xi, \xi \rangle \right), \qquad \xi \in E_{\mathbf{C}}.$$

Note that the right hand side is a "normalized" exponential function. The explicit form of $: x^{\otimes n} :$ is well known, see e.g., [**27**], [**28**]. We call ϕ_ξ an *exponential vector*.

Let H_n be the closed subspace of (L^2) spanned by the functions $x \mapsto \left\langle : x^{\otimes n} :, f \right\rangle$, where f runs over $E_{\mathbf{C}}^{\widehat{\otimes} n}$. Then, we have the so-called Wiener-Itô decomposition of (L^2) into an orthogonal direct sum of H_n. More precisely, for each $\phi \in (L^2)$ there exists a unique sequence $(f_n)_{n=0}^{\infty}$, $f_n \in H_{\mathbf{C}}^{\widehat{\otimes} n}$, such that

$$\phi(x) = \sum_{n=0}^{\infty} \left\langle : x^{\otimes n} :, f_n \right\rangle, \qquad x \in E^*, \tag{2-2}$$

where each $x \mapsto \left\langle : x^{\otimes n} :, f_n \right\rangle$ is a function in H_n and the series is an orthogonal direct sum. In that case

$$\|\phi\|_0^2 \equiv \int_{E^*} |\phi(x)|^2 \mu(dx) = \sum_{n=0}^{\infty} n! \, |f_n|_0^2 .$$

The above correspondence gives a unitary isomorphism between (L^2) and the Boson Fock space over $H_{\mathbf{C}}$.

2.3. Scalar-valued white noise functionals. We first define the second quantized operator $\Gamma(A)$ for A introduced in §2.1. Let $\phi \in (L^2)$ be given as in (2-2) and we put

$$\Gamma(A)\phi(x) = \sum_{n=0}^{\infty} \left\langle : x^{\otimes n} :, A^{\otimes n} f_n \right\rangle, \qquad \phi \in \mathrm{Dom}\,(\Gamma(A)).$$

Equipped with the maximal domain, $\Gamma(A)$ becomes a positive selfadjoint operator on (L^2). Let (E) be the standard CH-space constructed from the pair $((L^2), \Gamma(A))$. Since $\Gamma(A)$ admits Hilbert-Schmidt inverse by the hypothesis (S), (E) is a nuclear Fréchet space and we obtain a complex Gelfand triple:

$$(E) \subset (L^2) = L^2(E^*, \mu; \mathbf{C}) \subset (E)^*.$$

Elements in (E) and $(E)^*$ are called a *test (white noise) functional* and a *generalized (white noise) functional*, respectively. We denote by $\langle\!\langle \cdot , \cdot \rangle\!\rangle$ the canonical bilinear form on $(E)^* \times (E)$ and by $\|\cdot\|_p$ the norm introduced from $\Gamma(A)$, namely,

$$\|\phi\|_p^2 = \|\Gamma(A)^p \phi\|_0^2 = \sum_{n=0}^{\infty} n! \left|(A^{\otimes n})^p f_n\right|_0^2 = \sum_{n=0}^{\infty} n! \, |f_n|_p^2 , \tag{2-3}$$

where ϕ and $(f_n)_{n=0}^{\infty}$ are related as in (2-2). Therefore $\phi \in (L^2)$ belongs to (E) if and only if $f_n \in E_{\mathbf{C}}^{\widehat{\otimes} n}$ for all n and $\sum_{n=0}^{\infty} n! \, |f_n|_p^2 < \infty$ for all $p \geq 0$.

We use a similar (but formal) expression for a generalized white noise functional:

$$\Phi(x) = \sum_{n=0}^{\infty} \left\langle : x^{\otimes n} :, F_n \right\rangle, \tag{2-4}$$

where $F_n \in (E_{\mathbf{C}}^{\widehat{\otimes} n})_{\mathrm{sym}}^*$ and

$$\|\Phi\|_{-p}^2 = \sum_{n=0}^{\infty} n! \, |F_n|_{-p}^2 . \tag{2-5}$$

Note that (2-5) is valid for all $p \in \mathbf{R}$ though the value can be infinite. By construction $\|\Phi\|_{-p} < \infty$ for some $p \geq 0$. If Φ is given as above, for any $\phi \in (E)$ we have

$$\langle\!\langle \Phi , \phi \rangle\!\rangle = \sum_{n=0}^{\infty} n! \, \langle F_n , f_n \rangle ,$$

where $(f_n)_{n=0}^{\infty}$ is determined as in (2-2). It is also known that any $\Phi \in (E)^*$ is expressible in the above form.

3. INTEGRAL KERNEL OPERATORS

3.1. Hida's differential operator. For any $y \in E^*$ and $\phi \in (E)$ we put

$$D_y \phi(x) = \lim_{\theta \to 0} \frac{\phi(x + \theta y) - \phi(x)}{\theta}, \qquad x \in E^*. \tag{3-1}$$

It is known that the limit always exists and that $D_y \in \mathcal{L}((E),(E))$. Since the delta-functions δ_t are elements in E^* by hypotheses (H1)-(H3), we may define

$$\partial_t = D_{\delta_t}, \qquad t \in T.$$

This is called *Hida's differential operator.* Obviously, ∂_t is a rigorously defined *annihilation operator* at a point $t \in T$. It should be therefore emphasized that ∂_t is *not* an operator-valued distribution as in most literatures but a continuous operator for itself. The *creation operator* is by definition the adjoint $\partial_t^* \in \mathcal{L}((E)^*,(E)^*)$.

As is easily expected, ∂_t and ∂_t^* satisfy the so-called canonical commutation relation:

$$[\partial_s, \partial_t] = 0, \qquad [\partial_s^*, \partial_t^*] = 0, \qquad [\partial_s, \partial_t^*] = \delta_s(t) I, \qquad s, t \in T. \tag{3-2}$$

The last relation is understood in a generalized sense.

3.2. Integral kernel operators. For $\phi, \psi \in (E)$ let $\eta_{\phi,\psi}$ be a function on T^{l+m} defined by

$$\eta_{\phi,\psi}(s_1, \cdots, s_l, t_1, \cdots, t_m) = \left\langle\!\left\langle \partial_{s_1}^* \cdots \partial_{s_l}^* \partial_{t_1} \cdots \partial_{t_m} \phi , \psi \right\rangle\!\right\rangle . \tag{3-3}$$

Then one see that $\eta_{\phi,\psi} \in E_{\mathbf{C}}^{\otimes(l+m)}$, and that the bilinear form $\phi, \psi \mapsto \left\langle \kappa , \eta_{\phi,\psi} \right\rangle$ is continuous on $(E) \times (E)$ for any $\kappa \in (E_{\mathbf{C}}^{\otimes(l+m)})^*$. Thus a continuous linear operator $\Xi_{l,m}(\kappa) \in \mathcal{L}((E),(E)^*)$ is determined uniquely by

$$\left\langle\!\left\langle \Xi_{l,m}(\kappa)\phi , \psi \right\rangle\!\right\rangle = \left\langle \kappa , \eta_{\phi,\psi} \right\rangle , \qquad \phi, \psi \in (E). \tag{3-4}$$

In other words, $\Xi_{l,m}(\kappa)$ is defined through two canonical bilinear forms:

$$\langle\langle \Xi_{l,m}(\kappa)\phi, \psi\rangle\rangle = \left\langle \kappa, \langle\langle \partial_{s_1}^* \cdots \partial_{s_l}^* \partial_{t_1} \cdots \partial_{t_m}\phi, \psi\rangle\rangle\right\rangle, \qquad \phi, \psi \in (E).$$

This suggests us to employ a formal integral expression as in (0-1). We call $\Xi_{l,m}(\kappa)$ an *integral kernel operator* with *kernel distribution* κ. But we must not forget that $\Xi_{l,m}(\kappa)$ becomes a continuous operator in $\mathcal{L}((E),(E)^*)$ for any $\kappa \in (E_{\mathbf{C}}^{\otimes(l+m)})^*$. In view of a precise norm estimate of $\Xi_{l,m}(\kappa)\phi$, $\phi \in (E)$, we see that $\Xi_{l,m}(\kappa) \in \mathcal{L}((E),(E))$ if and only if $\kappa \in (E_{\mathbf{C}}^{\otimes l}) \otimes (E_{\mathbf{C}}^{\otimes m})^*$.

The kernel distribution is not uniquely determined due to the fact (3-2). For the uniqueness we only need to consider the space $(E_{\mathbf{C}}^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$ of all $\kappa \in (E_{\mathbf{C}}^{\otimes(l+m)})^*$ which is symmetric with respect to the first l and the last m variables independently.

4. FOCK EXPANSION

4.1. Symbol of operators. Since the exponential vectors $\{\phi_\xi;\ \xi \in E_{\mathbf{C}}\}$ spans a dense subspace of (E), the behavior of an operator $\Xi \in \mathcal{L}((E),(E)^*)$ on those vectors is worthwhile to study. For $\Xi \in \mathcal{L}((E),(E)^*)$ a function on $E_{\mathbf{C}} \times E_{\mathbf{C}}$ defined by

$$\widehat{\Xi}(\xi,\eta) = \langle\langle \Xi\phi_\xi, \phi_\eta\rangle\rangle, \qquad \xi,\eta \in E_{\mathbf{C}}, \tag{4-1}$$

is called the *symbol* of Ξ after Berezin **[3]** and Krée-Rączka **[13]**. For example,

$$\Xi_{l,m}(\kappa)\widehat{\ }(\xi,\eta) = \left\langle \kappa, \eta^{\otimes l} \otimes \xi^{\otimes m}\right\rangle e^{\langle \xi,\eta\rangle}, \qquad \xi,\eta \in E_{\mathbf{C}}, \quad \kappa \in E_{\mathbf{C}}^{\otimes(l+m)}. \tag{4-2}$$

In particular,

$$\widehat{\partial_t}(\xi,\eta) = \xi(t)e^{\langle \xi,\eta\rangle}, \qquad \widehat{\partial_t^*}(\xi,\eta) = \eta(t)e^{\langle \xi,\eta\rangle}, \qquad \xi,\eta \in E_{\mathbf{C}}.$$

We then observe important propertirs of $\Theta = \widehat{\Xi}$. For any $\Xi \in \mathcal{L}((E),(E)^*)$ we have

(O1) (analyticity) For any $\xi, \xi_1, \eta, \eta_1 \in E_{\mathbf{C}}$, the function

$$z, w \mapsto \Theta(z\xi + \xi_1, w\eta + \eta_1), \qquad z, w \in \mathbf{C},$$

is entire holomorphic;

(O2) (boundedness) There exist constant numbers $C \geq 0$, $K \geq 0$ and $p \in \mathbf{R}$ such that

$$|\Theta(\xi,\eta)| \leq C \exp K\left(|\xi|_p^2 + |\eta|_p^2\right), \qquad \xi,\eta \in E_{\mathbf{C}}.$$

If $\Xi \in \mathcal{L}((E),(E))$, we have a stronger estimate:

(O2′) (boundedness) For any $p \geq 0$ and $\epsilon > 0$ there exist $C \geq 0$ and $q \geq 0$ such that

$$|\Theta(\xi,\eta)| \leq C \exp \epsilon \left(|\xi|_{p+q}^2 + |\eta|_{-p}^2 \right), \qquad \xi,\eta \in E_{\mathbf{C}}.$$

More important is that the above listed properties reproduce the operator on white noise functionals.

THEOREM 4.1. *Assume that a* $\mathbf{C}$*-valued function* Θ *on* $E_{\mathbf{C}} \times E_{\mathbf{C}}$ *satisfies the conditions (O1) and (O2). Then, there exists a unique family of kernel distributions* $(\kappa_{l,m})_{l,m=0}^{\infty}$, $\kappa_{l,m} \in (E_{\mathbf{C}}^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$, *such that*

$$\Theta(\xi,\eta) = \sum_{l,m=0}^{\infty} \left\langle\!\left\langle \Xi_{l,m}(\kappa_{l,m})\phi_\xi \,, \phi_\eta \right\rangle\!\right\rangle, \qquad \xi,\eta \in E_{\mathbf{C}}. \tag{4-3}$$

Moreover, the series

$$\Xi\phi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})\phi, \qquad \phi \in (E), \tag{4-4}$$

converges in $(E)^*$, $\Xi \in \mathcal{L}((E),(E)^*)$ *and* $\widehat{\Xi} = \Theta$. *If* Θ *satisfies the conditions (O1) and (O2′), the kernel distribution* $\kappa_{l,m}$ *belongs to* $\left((E_{\mathbf{C}}^{\otimes l}) \otimes (E_{\mathbf{C}}^{\otimes m})^*\right)_{\mathrm{sym}(l,m)} = (E_{\mathbf{C}}^{\widehat{\otimes} l}) \otimes (E_{\mathbf{C}}^{\otimes m})^*_{\mathrm{sym}}$. *In that case, the series (4-4) converges in* (E) *and* $\Xi \in \mathcal{L}((E),(E))$.

The idea of the proof is as follows. If (4-3) holds, we see from (4-2) that

$$e^{-\langle \xi,\eta\rangle}\Theta(\xi,\eta) = \sum_{l,m=0}^{\infty} \left\langle \kappa_{l,m} \,, \eta^{\otimes l} \otimes \xi^{\otimes m} \right\rangle, \qquad \xi,\eta \in E_{\mathbf{C}}. \tag{4-5}$$

Therefore, an $(l+m)$-linear form $\kappa_{l,m}$ on $E_{\mathbf{C}}^{\otimes(l+m)}$ is obtained by Taylor expansion of $e^{-\langle \xi,\eta\rangle}\Theta(\xi,\eta)$. We then use the assumptions to obtain a norm estimate of $\kappa_{l,m}$ which implies that $\kappa_{l,m} \in (E_{\mathbf{C}}^{\otimes(l+m)})^*$. Finally with the help of precise norm estimate of $\Xi_{l,m}(\kappa)\phi$ we prove the strong convergence of (4-4) in $(E)^*$ or (E). For the complete proof see **[26]** and **[28]**.

COROLLARY 4.2. *Let* Θ *be a function on* $E_{\mathbf{C}} \times E_{\mathbf{C}}$ *with values in* $\mathbf{C}$. *Then, there exists* $\Xi \in \mathcal{L}((E),(E)^*)$ *with* $\Theta = \widehat{\Xi}$ *if and only if* Θ *satisfies (O1) and (O2). Furthermore, there exists* $\Xi \in \mathcal{L}((E),(E))$ *with* $\Theta = \widehat{\Xi}$ *if and only if* Θ *satisfies (O1) and (O2′).*

In some practical problems operators on Fock space are only defined on the exponential vectors $\{\phi_\xi;\ \xi \in E_{\mathbf{C}}\}$ due to the fact that they are linearly independent. The above result is therefore useful for checking whether the operator comes into our framework.

4.2. Fock expansion. Since the symbol of $\Xi \in \mathcal{L}((E),(E)^*)$ satisfies (O1) and (O2), Theorem 4.1 gives rise to reconstruction of Ξ in terms of integral kernel operators. A similar discussion for $\Xi \in \mathcal{L}((E),(E))$ is also valid and we come to the following

THEOREM 4.3. *For any $\Xi \in \mathcal{L}((E),(E)^*)$ there exists a unique family of kernel distributions $(\kappa_{l,m})_{l,m=0}^{\infty}$, $\kappa_{l,m} \in (E_{\mathbf{C}}^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$, such that the series (4-4) converges in $(E)^*$. If $\Xi \in \mathcal{L}((E),(E))$, then every kernel distribution $\kappa_{l,m}$ belongs to $\left((E_{\mathbf{C}}^{\otimes l}) \otimes (E_{\mathbf{C}}^{\otimes m})^*\right)_{\mathrm{sym}(l,m)} = (E_{\mathbf{C}}^{\widehat{\otimes} l}) \otimes (E_{\mathbf{C}}^{\otimes m})^*_{\mathrm{sym}}$ and (4-4) converges in (E).*

The unique expression of $\Xi \in \mathcal{L}((E),(E)^*)$ given as in (4-4) is called the *Fock expansion* of Ξ. As we explained in §4.1, for a given $\Xi \in \mathcal{L}((E),(E)^*)$ the kernel distributions $(\kappa_{l,m})_{l,m=0}^{\infty}$ are obtained from the Taylor expansion of $e^{-\langle \xi,\eta\rangle}\widehat{\Xi}(\xi,\eta)$, see (4-5).

Since every bounded operator on (L^2) belongs to $\mathcal{L}((E),(E)^*)$, it admits the Fock expansion. However, the convergence can not be discussed within the framework of Hilbert space. In fact, except scalar operators no integral kernel operator admits an extension to a bounded operator on (L^2). Therefore, the Fock expansion of a non-scalar bounded operator on (L^2) is always an infinite series of unbounded operators.

4.3. Examples of Fock expansion.

EXAMPLE 1 (differential operators and translations). For $y \in E^*$ a differential operator D_y is defined by (3-1). Then

$$D_y = \Xi_{0,1}(y), \qquad D_y^* = \Xi_{1,0}(y).$$

In particular, for $t \in T$ it holds that

$$\partial_t = \Xi_{0,1}(\delta_t), \qquad \partial_t^* = \Xi_{1,0}(\delta_t).$$

As is easily expected, D_y is related to a translation operator. For $y \in E^*$ we define

$$T_y\phi(x) = \phi(x+y), \qquad x \in E^*, \qquad \phi \in (E).$$

Then $T_y \in \mathcal{L}((E),(E))$ and

$$T_y = \sum_{n=0}^{\infty} \frac{1}{n!}\Xi_{0,n}(y^{\otimes n}) = \sum_{n=0}^{\infty} \frac{1}{n!} D_y^n.$$

Note also that the Fock expansion of T_y above yields Taylor expansion of test white noise functionals obtained by Potthoff-Yan [**32**].

EXAMPLE 2 (multiplication operators). As is well known (see e.g., [**14**]), the pointwise multiplication induces a continuous bilinear map from $(E) \times (E)$ into (E). Hence each $\Phi \in (E)^*$ gives rise to a continuous operator in $\mathcal{L}((E),(E)^*)$ in such a way that $\langle\langle \Phi\phi , \psi \rangle\rangle = \langle\langle \Phi , \phi\psi \rangle\rangle$, $\phi, \psi \in (E)$. If Φ is given as $\Phi(x) = \sum_{n=0}^{\infty} \left\langle : x^{\otimes n} : , F_n \right\rangle$ in a symbolic sense (see (2-3)), we have

$$\Phi = \sum_{l,m=0}^{\infty} \binom{l+m}{m} \Xi_{l,m}(F_{l+m}).$$

In particular, for $x(t) = \left\langle : x^{\otimes 1} : , \delta_t \right\rangle$, $t \in T$, we have

$$x(t) = \Xi_{1,0}(\delta_t) + \Xi_{0,1}(\delta_t) = \partial_t^* + \partial_t.$$

This is an integrand of quantum Brownian motion (see e.g., [**11**]), and therefore may be called *quantum white noise.*

EXAMPLE 3 (Laplacians). Let $\tau \in (E \otimes E)^*$ be defined as $\langle \tau , \xi \otimes \eta \rangle$, $\xi, \eta \in E$. The integral kernel operators with τ being the kernel distribution are of great importance. We put

$$\Delta_G = \Xi_{0,2}(\tau) = \int_{T \times T} \tau(s,t) \partial_s \partial_t ds dt, \quad N = \Xi_{1,1}(\tau) = \int_{T \times T} \tau(s,t) \partial_s^* \partial_t ds dt.$$

These are called the *Gross Laplacian* and the *number operator*, respectively. It is noted that both are continuous operators from (E) into itself. Obviously, N^* is an extension of N and Δ_G^* is given as $\Delta_G^* = \Xi_{2,0}(\tau)$.

A white noise analogue of the Euclidean norm is $\left\langle : x^{\otimes 2} : , \tau \right\rangle$. Regarded as a multiplication operator,

$$\left\langle : x^{\otimes 2} : , \tau \right\rangle = \Delta_G^* + 2N + \Delta_G.$$

The above mentioned Laplacians are characterized as rotation-invariant operators on white noise functionals. For the full description, see [**24**].

EXAMPLE 4 (projection onto the n-th chaos). Let $(L^2) = \sum_{n=0}^{\infty} \oplus \mathcal{H}_n$ be the Wiener-Itô decomposition (see also §2.2) and let π_n be the projection onto the n-th chaos $\mathcal{H}_n$. It is easy to see that $\pi_n \in \mathcal{L}((E),(E))$ if restricted to (E). Then,

$$\pi_n = \sum_{l=n}^{\infty} \frac{(-1)^{l-n}}{(l-n)!n!} \Xi_{l,l}(\lambda_l),$$

where

$$\lambda_l = \sum_{i_1, \cdots, i_l = 0}^{\infty} e_{i_1} \otimes \cdots \otimes e_{i_l} \otimes e_{i_1} \otimes \cdots \otimes e_{i_l} \in (E^{\otimes 2l})^*. \tag{4-6}$$

It is also interesting to note that

$$\Xi_{l,l}(\lambda_l) = N(N-1) \cdots (N-l+1),$$

where N is the number operator, see Example 3.

EXAMPLE 5 (Fourier-Wiener transfrom). Let $(\exp(i\theta N))_{\theta \in \mathbf{R}}$ be a one-parameter group of Fourier-Wiener transform, namely, it is a one-parameter group of unitary operators on

(L^2) with the number operator N being the infinitesimal generator. Then $\exp(i\theta N) \in \mathcal{L}((E),(E))$ and, with λ_l defined as in (4-6) we have

$$\exp(i\theta N) = \sum_{l=0}^{\infty} \frac{(e^{i\theta}-1)^l}{l!} \Xi_{l,l}(\lambda_l).$$

EXAMPLE 6 (Weyl form of canonical commutation relation). We consider representations of the additive group E. For $\xi \in E$ and $\phi \in (E)$ put

$$\begin{aligned} P_\xi \phi(x) &= \phi(x+\xi) \exp\left(-\frac{1}{2}\langle x, \xi\rangle - \frac{1}{4}\langle \xi, \xi\rangle\right), \\ Q_\xi \phi(x) &= e^{i\langle x, \xi\rangle}\phi(x). \end{aligned}$$

Then, P_ξ and Q_ξ belong to $\mathcal{L}((E),(E))$ and are extended to unitary operators on (L^2). The Fock expansions of P_ξ and Q_ξ are given as

$$\begin{aligned} P_\xi &= e^{-\langle \xi, \xi\rangle/8} \sum_{l,m=0}^{\infty} \frac{(-1)^l}{l!m!}\left(\frac{1}{2}\right)^{l+m} \Xi_{l,m}(\xi^{\otimes(l+m)}), \\ Q_\xi &= e^{-\langle \xi, \xi\rangle/2} \sum_{l,m=0}^{\infty} \frac{i^{l+m}}{l!m!} \Xi_{l,m}(\xi^{\otimes(l+m)}). \end{aligned}$$

Moreover,

$$p_\xi \equiv \frac{d}{d\theta} P_{\theta\xi}\bigg|_{\theta=0} = \frac{1}{2}\left(D_\xi - D_\xi^*\right), \qquad q_\xi \equiv \frac{d}{d\theta} Q_{\theta\xi}\bigg|_{\theta=0} = i\left(D_\xi + D_\xi^*\right).$$

These operators belong to $\mathcal{L}((E),(E))$ again and satisfy the canonical commutation relation.

EXAMPLE 7 (Kuo's Fourier transform). There is a white noise analogue of a usual Fourier transform on $\mathbf{R}^n$ introduced by Kuo in **[16]** at somehow formal level and in **[18]** rigorously. This operator is characterized as a unique operator in $\mathcal{L}((E)^*,(E)^*)$ which intertwines differential operators and coordinate multiplication operators, see **[8]**. Moreover, Kuo's Fourier transform is imbedded in a one-parameter group of transformations called *Fourier-Mehler transforms* $(\mathfrak{F}_\theta)_{\theta\in\mathbf{R}} \subset \mathcal{L}((E)^*,(E)^*)$ in such a way that $\mathfrak{F}_{-\pi/2} = \mathfrak{F}$, see **[17]**, **[18]**. Regarded as an operator in $\mathcal{L}((E),(E)^*)$, $\mathfrak{F}_\theta$ admits Fock expansion:

$$\mathfrak{F}_\theta = \sum_{l,m=0}^{\infty} \frac{1}{l!m!}\left(\frac{i}{2}e^{i\theta}\sin\theta\right)^l \left(e^{i\theta}-1\right)^m \Xi_{2l+m,m}(\tau^{\otimes l}\otimes\lambda_m).$$

EXAMPLE 8 (integral-sum kernel operators). In order to describe solutions of certain quantum stochastic differential equations Maassen **[21]** introduced a certain class of operators on Fock space. The discussion has been improved by Lindsay [20] and Meyer [22], and those operators are now called *integral-sum kernel operators.* Formal relation between their operators and our integral kernel operators is discussed in [25] and [26].

5. CALCULUS ON VECTOR-VALUED FUNCTIONALS

5.1. Construction. Let $\mathcal{H}$ be another complex Hilbert space whose norm is denoted by $|\cdot|_0$ again. This will be called an *initial space.* Let B be a positive selfadjoint operator on $\mathcal{H}$ with $\inf \operatorname{Spec}(B) > 0$ and let $\mathcal{E}$ be the standard CH-space constructed from $(\mathcal{H}, B)$. Then $(E) \otimes \mathcal{E}$ is the space of *$\mathcal{E}$-valued test white noise functionals* and its dual space $((E) \otimes \mathcal{E})^* = (E)^* \otimes \mathcal{E}^*$ consists of *$\mathcal{E}^*$-valued generalized white noise functionals.* It must be recalled here that (E) is a nuclear Fréchet space. We put

$$\sigma = (\inf \operatorname{Spec}(B))^{-1} = \left\| B^{-1} \right\|_{\mathrm{OP}} > 0. \tag{5-1}$$

Note that the identity operator on $\mathcal{H}$ can be taken for B. In that case, identifying $\mathcal{H}^*$ with $\mathcal{H}$, we obtain $\mathcal{H}$-valued test and generalized white noise functionals. Since (E) is the standard CH-space constructed from $(\Gamma(A), (L^2))$, we have

THEOREM 5.1. *Notations and assumptions being as above, $\Gamma(A) \otimes B$ is a positive selfadjoint operator on $L^2(E^*, \mu) \otimes \mathcal{H} \cong L^2(E^*, \mu; \mathcal{H})$ with $\inf \operatorname{Spec}(\Gamma(A) \otimes B) > 0$. The standard CH-space constructed from $(L^2(E^*, \mu) \otimes \mathcal{H}, \Gamma(A) \otimes B)$ is isomorphic to $(E) \otimes \mathcal{E}$.*

By virtue of the above fact we may extend basic notations for scalar-valued functionals to the case of vector-valued ones. The canonical bilinear form on $((E) \otimes \mathcal{E})^* \times ((E) \otimes \mathcal{E})$ is denoted by $\langle\langle \cdot , \cdot \rangle\rangle$ again. Similarly, the norms of $(E) \otimes \mathcal{E}$ are denoted again by $\|\cdot\|_p$. While, the bilinear form $\langle \cdot , \cdot \rangle$ on $(E_{\mathbb{C}}^{\otimes n})^* \times E_{\mathbb{C}}^{\otimes n}$ is extended to a continuous bilinear map from $(E_{\mathbb{C}}^{\otimes n})^* \times (E_{\mathbb{C}}^{\otimes n} \otimes \mathcal{E})$ into $\mathcal{E}$ in an obvious way. Then, according to the Wiener-Itô decomposition :

$$L^2(E^*, \mu; \mathcal{H}) \cong (L^2) \otimes \mathcal{H} = \sum_{n=0}^{\infty} \oplus (H_n \otimes \mathcal{H}),$$

we adopt formally the same notations for $\mathcal{E}$-valued test functionals and $\mathcal{E}^*$-valued generalized functionals as in (2-2) and (2-4), respectively. However, $f_n \in E_{\mathbb{C}}^{\widehat{\otimes} n} \otimes \mathcal{E}$ and $F_n \in (E_{\mathbb{C}}^{\otimes n} \otimes \mathcal{E})^*_{\mathrm{sym}} \equiv (E_{\mathbb{C}}^{\otimes n})^*_{\mathrm{sym}} \otimes \mathcal{E}^*$. Note also that (2-3) and (2-5) remain valid.

5.2. Continuous version theorem. Kubo-Yokoi's continuous version theorem [15] is generalized to the case of vector-valued functionals.

THEOREM 5.2. *For each $\phi \in (E) \otimes \mathcal{E}$ there exists a unique continuous function $\widetilde{\phi} : E^* \to \mathcal{E}$ such that $\phi(x) = \widetilde{\phi}(x)$ for μ-a.e. $x \in E^*$.*

The proof requires an explicit estimate of $|\phi_0(x) - \phi_0(y)|$, $\phi_0 \in (E)$, $x, y \in E^*$, in terms of a defining seminorm of E^* and a general result on the π-tensor product.

The white noise delta function $\delta_x \in (E)^*$ now becomes a continuous linear map from $(E) \otimes \mathcal{E}$ into $\mathcal{E}$, i.e., $\delta_x \in \mathcal{L}((E) \otimes \mathcal{E}, \mathcal{E})$. Therefore the convergence in $(E) \otimes \mathcal{E}$ implies the pointwise convergence as $\mathcal{E}$-valued functions on E^*.

5.3. The S-transform. The S-transform for scalar-valued functionals due to Kubo-Takenaka **[14]** is naturally extended to the case of vector-valued white noise functionals. The *S-transform* of $\Phi \in ((E) \otimes \mathcal{E})^*$ is a function on $E_{\mathbf{C}}$ with values in $\mathcal{E}^*$ defined by

$$\langle S\Phi(\xi), u\rangle = \langle\!\langle \Phi, \phi_\xi \otimes u \rangle\!\rangle, \qquad u \in \mathcal{E}, \quad \xi \in E_{\mathbf{C}}. \tag{5-2}$$

Then $F = S\Phi : E_{\mathbf{C}} \to \mathcal{E}^*$ satisfies the following properties:

(F1) (analyticity) For any fixed $\xi, \xi_1 \in E_{\mathbf{C}}$ and $u \in \mathcal{E}$, the function

$$z \mapsto \langle F(z\xi + \xi_1), u\rangle, \qquad z \in \mathbf{C},$$

is entire holomorphic;

(F2) (boundedness) There exist $C \geq 0$, $K \geq 0$ and $p \geq 0$ such that

$$|\langle F(\xi), u\rangle| \leq C\,|u|_p \exp\left(K\,|\xi|_p^2\right), \qquad \xi \in E_{\mathbf{C}}, \quad u \in \mathcal{E}.$$

If $\Phi = \phi \in (E) \otimes \mathcal{E}$, then $F = S\phi$ satisfies a stronger estimate:

(F2$'$) (boundedness) For any $\epsilon > 0$ and $p \geq 0$ there exists $C \geq 0$ such that

$$|\langle F(\xi), u\rangle| \leq C\,|u|_{-p} \exp\left(\epsilon\,|\xi|_{-p}^2\right), \qquad \xi \in E_{\mathbf{C}}, \quad u \in \mathcal{E}.$$

Note also that if a function $F : E_{\mathbf{C}} \to \mathcal{E}^*$ satisfies (F2$'$), then $F(\xi) \in \mathcal{E}$ for all $\xi \in E_{\mathbf{C}}$.

More important is that the above listed properties characterize the S-transform of generalized or test functionals. Namely, characterization theorem due to Potthoff-Streit **[31]** and Kuo-Potthoff-Streit **[19]** holds for vector-valued functionals as well.

THEOREM 5.3. *If a function $F : E_{\mathbf{C}} \to \mathcal{E}^*$ satisfies (F1) and (F2), there exists $\Phi \in ((E) \otimes \mathcal{E})^*$ such that $S\Phi = F$. If F satisfies (F1) and (F2$'$), there exists $\phi \in (E) \otimes \mathcal{E}$ such that $S\phi = F$.*

For the proof we need the original result on scalar-valued functionals with precise norm estimate obtained in **[28]** and the famous kernel theorem for nuclear spaces. (Recall that $\mathcal{E}$ is not necessarily nuclear; The nuclearity of (E) is essential.)

5.4. Contraction of tensor products. In order to define an integral kernel operator on vector-valued functionals we do not follow the method used in §3.2 but adopt a more direct definition. (This is also applicable to the case of scalar-valued functionals.)

Let $f \in E_{\mathbf{C}}^{\otimes(l+m)}$ and $g \in E_{\mathbf{C}}^{\otimes(n+m)}$ be given in Fourier series expansions:

$$f = \sum_{\mathbf{i},\mathbf{j}} \langle f \, , e(\mathbf{i}) \otimes e(\mathbf{j}) \rangle \, e(\mathbf{i}) \otimes e(\mathbf{j}), \qquad g = \sum_{\mathbf{k},\mathbf{j}} \langle g \, , e(\mathbf{k}) \otimes e(\mathbf{j}) \rangle \, e(\mathbf{k}) \otimes e(\mathbf{j}),$$

where

$$e(\mathbf{i}) = e_{i_1} \otimes \cdots \otimes e_{i_l}, \quad e(\mathbf{j}) = e_{j_1} \otimes \cdots \otimes e_{j_m}, \quad e(\mathbf{k}) = e_{k_1} \otimes \cdots \otimes e_{k_n}.$$

We then put

$$f \otimes_m g = \sum_{\mathbf{i},\mathbf{k}} \left(\sum_{\mathbf{j}} \langle f \, , e(\mathbf{i}) \otimes e(\mathbf{j}) \rangle \, \langle g \, , e(\mathbf{k}) \otimes e(\mathbf{j}) \rangle \right) e(\mathbf{i}) \otimes e(\mathbf{k}). \tag{5-3}$$

This is a contraction of tensor product. It is important to have its precise norm estimate. Define $|f|_{l,m;p,r}$ by

$$|f|^2_{l,m;p,r} = \sum_{\mathbf{i},\mathbf{j}} |\, \langle f \, , e(\mathbf{i}) \otimes e(\mathbf{j}) \rangle \,|^2 \, |e(\mathbf{i})|_p^2 \, |e(\mathbf{j})|_r^2 , \qquad p, r \in \mathbf{R}. \tag{5-4}$$

With these notation a simple application of the Schwartz inequality yields

$$|f \otimes_m g|_{l,n;p,q} \le |f|_{l,m;p,r} \, |g|_{n,m;q,-r} , \qquad f \in E_{\mathbf{C}}^{\otimes(l+m)}, g \in E_{\mathbf{C}}^{\otimes(n+m)}, p, q, r \in \mathbf{R}. \tag{5-5}$$

We next generalize the contraction (5-3) to the vector-valued case, namely, for $\kappa \in \mathcal{L}(E_{\mathbf{C}}^{\otimes(l+m)}, \mathcal{L}(\mathcal{E}, \mathcal{E}^*))$ and $f \in E_{\mathbf{C}}^{\otimes(m+n)} \otimes \mathcal{E}$ we shall define $\kappa \otimes_m f \in (E_{\mathbf{C}}^{\otimes(l+m)} \otimes \mathcal{E})^*$. For that purpose we need to define some norms. For a linear map $\kappa : E_{\mathbf{C}}^{\otimes(l+m)} \to \mathcal{L}(\mathcal{E}, \mathcal{E}^*)$ and $p, q, r, s \in \mathbf{R}$ we put

$$\|\kappa\|_{l,m;p,q;r,s} = \sup \left\{ \sum_{\mathbf{i},\mathbf{j}} |\langle \kappa(e(\mathbf{i}) \otimes e(\mathbf{j}))u \, , v \rangle|^2 \, |e(\mathbf{i})|_p^2 \, |e(\mathbf{j})|_q^2 \, ; \quad \begin{array}{l} |u|_{-s} \le 1 \\ |v|_{-r} \le 1 \end{array} \right\}^{1/2}$$

For brevity we put $\|\kappa\|_p = \|\kappa\|_{l,m;p,p;p,p}$. It can be proved that $\kappa \in \mathcal{L}(E_{\mathbf{C}}^{\otimes(l+m)}, \mathcal{L}(\mathcal{E}, \mathcal{E}^*))$ if and only if $\|\kappa\|_{-p} < \infty$ for some $p \ge 0$.

Let $\kappa \in \mathcal{L}(E_{\mathbf{C}}^{\otimes(l+m)}, \mathcal{L}(\mathcal{E}, \mathcal{E}^*))$ and $f \in E_{\mathbf{C}}^{\widehat{\otimes}(m+n)} \otimes \mathcal{E}$. It then follows from a general theory that there exists an element $\kappa \otimes_m f \in (E_{\mathbf{C}}^{\otimes(l+m)} \otimes \mathcal{E})^*$ uniquely determined by

$$\langle \kappa \otimes_m (f_0 \otimes u) \, , g_0 \otimes v \rangle = \langle \kappa(g_0 \otimes_n f_0)u \, , v \rangle , \quad f_0 \in E_{\mathbf{C}}^{\otimes(m+n)}, g_0 \in E_{\mathbf{C}}^{\otimes(l+n)}, \quad u, v \in \mathcal{E}.$$

Moreover, by a direct calculation with (5-5) we obtain

$$(5\text{-}6)\qquad |\kappa \otimes_m f|_{-(p+1)} \le \delta^{l+m+2n} \sigma \,\|\kappa\|_{l,m;-p,-q;-p,-s} \, |f|_{m,n;q+1,-p+1;s} \,,$$

where

$$|f|_{m,n;q+1,-p+1;s} = \left| \left((A^{\otimes m})^{q+1} \otimes (A^{\otimes n})^{-p+1} \otimes B^s \right) f \right|_0 .$$

5.5. Integral kernel operators. With each $\kappa \in \mathcal{L}(E_{\mathbf{C}}^{\otimes(l+m)}, \mathcal{L}(\mathcal{E}, \mathcal{E}^*))$ we shall associate a continuous operator $\Xi_{l,m}(\kappa)$ from $(E) \otimes \mathcal{E}$ into $((E) \otimes \mathcal{E})^*$. For $\phi \in (E) \otimes \mathcal{E}$ in the canonical form:

$$\phi(x) = \sum_{n=0}^{\infty} \left\langle : x^{\otimes n} : , f_n \right\rangle ,$$

where $f_n \in E_{\mathbf{C}}^{\otimes n} \otimes \mathcal{E}$, we put

$$\Xi_{l,m}(\kappa)\phi(x) = \sum_{n=0}^{\infty} \frac{(n+m)!}{n!} \left\langle : x^{\otimes(l+n)} : , \kappa \otimes_m f_{n+m} \right\rangle .$$

Then a computation with (5-6) leads us to the following key result.

PROPOSITION 5.4. *Let $\kappa \in \mathcal{L}(E_{\mathbf{C}}^{\otimes(l+m)}, \mathcal{L}(\mathcal{E}, \mathcal{E}^*))$. Then, for any $\phi \in (E) \otimes \mathcal{E}$,*

$$\left\| \Xi_{l,m}(\kappa)\phi \right\|_p \le \rho^{-q/2} \delta^{l+m-1} \sigma^2 \left(l^l m^m \right)^{1/2} \left(\frac{\rho^{-\alpha/2}}{-\alpha e \log \rho} \right)^{l/2}$$
$$\times \left(\frac{\rho^{-\beta/2}}{-\beta e \log \rho} \right)^{m/2} \|\kappa\|_{l,m;p+1,-(p+q+1);p+1,-(p+q+1)} \, \|\phi\|_{p+q+2} \,,$$

whenever $p \in \mathbf{R}$, $q \ge 0$, $\alpha, \beta > 0$ satisfy

$$\|\kappa\|_{l,m;p+1,-(p+q+1);p+1,-(p+q+1)} < \infty, \qquad \delta^2 \rho^q < 1, \qquad \delta^4 \rho^{2q} \le \rho^{\alpha+\beta}.$$

Specializing the parameters p, q, α, β in Proposition 5.4, we come to the following

THEOREM 5.5. *For any $\kappa \in \mathcal{L}(E_{\mathbf{C}}^{\otimes(l+m)}, \mathcal{L}(\mathcal{E}, \mathcal{E}^*))$ the operator $\Xi_{l,m}(\kappa)$ becomes a continuous operator in $\mathcal{L}((E) \otimes \mathcal{E}, ((E) \otimes \mathcal{E})^*)$. Moreover,*

$$\left\| \Xi_{l,m}(\kappa)\phi \right\|_{-(p+1)} \le \delta^{l+m} \sigma \,\|\kappa\|_{-p} \, \|\phi\|_{p+1} \, (\delta\rho^p)^{-1} \left(l^l m^m \right)^{1/2} \left(\frac{(\delta\rho^p)^{-1}}{-2e \log \delta\rho^p} \right)^{(l+m)/2}$$

THEOREM 5.6. *For $\kappa \in \mathcal{L}(E_{\mathbf{C}}^{\otimes(l+m)}, \mathcal{L}(\mathcal{E}, \mathcal{E}^*))$ the following four conditions are equivalent:*

(i) *$\Xi_{l,m}(\kappa) \in \mathcal{L}((E) \otimes \mathcal{E}, (E) \otimes \mathcal{E})$;*
(ii) *for any $p \ge 0$ there exists $q \ge 0$ such that $|\kappa|_{l,m;p,-(p+q);p,-(p+q)} < \infty$;*
(iii) *for any $p \ge 0$ there exists $r, s \in \mathbf{R}$ such that $|\kappa|_{l,m;p,r;p,s} < \infty$;*
(iv) *$(\xi, \eta) \mapsto \kappa(\xi \otimes \eta)$ admits an extension to a separately continuous bilinear map from $(E_{\mathbf{C}}^{\otimes l})^* \times (E_{\mathbf{C}}^{\otimes m})$ into $\mathcal{L}(\mathcal{E}, \mathcal{E})$.*

Let $\mathcal{B}((E_{\mathbf{C}}^{\otimes l})^*, E_{\mathbf{C}}^{\otimes m}; \mathcal{L}(\mathcal{E},\mathcal{E}))$ denote the space of $\kappa \in \mathcal{L}(E_{\mathbf{C}}^{\otimes(l+m)}, \mathcal{L}(\mathcal{E},\mathcal{E}^*))$ satisfying one of the conditions in Theorem 5.6.

As is easily verified, the operator $\Xi_{l,m}(\kappa)$ is uniquely determined by

$$\langle\!\langle \Xi(\phi\otimes u), \psi\otimes v\rangle\!\rangle = \left\langle \kappa(\eta_{\phi,\psi})u, v\right\rangle, \qquad \phi,\psi\in(E), \quad u,v\in\mathcal{E},$$

where $\eta_{\phi,\psi}$ is defined as in (3-3). In view of (3-4) we again adopt a formal expression for $\Xi_{l,m}(\kappa)$ as in (0-1), where ∂_s^* and ∂_t are respectively shortened notation for $(\partial_s\otimes I)^*$ and $\partial_t\otimes I$, I being the identity operator on $\mathcal{E}$. While, $\kappa\in\mathcal{L}(E_{\mathbf{C}}^{\otimes(l+m)}, \mathcal{L}(\mathcal{E},\mathcal{E}^*))$ might be called a $\mathcal{L}(\mathcal{E},\mathcal{E}^*)$-valued distribution on T^{l+m}. The uniqueness of a kernel distribution is described by $\mathcal{L}(E_{\mathbf{C}}^{\otimes(l+m)}, \mathcal{L}(\mathcal{E},\mathcal{E}^*))_{\mathrm{sym}(l,m)}$ and $\mathcal{B}((E_{\mathbf{C}}^{\otimes l})^*, E_{\mathbf{C}}^{\otimes m}; \mathcal{L}(\mathcal{E},\mathcal{E}))_{\mathrm{sym}(l,m)}$ of which definitions are apparent.

5.6. Symbol of operators on vector-valued functionals. The *symbol* of $\Xi\in\mathcal{L}((E)\otimes\mathcal{E}, ((E)\otimes\mathcal{E})^*)$ is a function on $E_{\mathbf{C}}\times E_{\mathbf{C}}$ with values in $\mathcal{L}(\mathcal{E},\mathcal{E}^*)$ defined by

$$\left\langle \widehat{\Xi}(\xi,\eta)u, v\right\rangle = \left\langle\!\left\langle \Xi(\phi_\xi\otimes u), \phi_\eta\otimes v\right\rangle\!\right\rangle, \qquad \xi,\eta\in E_{\mathbf{C}}.$$

This is a direct generalization of an operator symbol introduced in §4.1. For an integral kernel operator we have

$$\Xi_{l,m}(\kappa)\widehat{\ }(\xi,\eta) = e^{\langle\xi,\eta\rangle}\kappa\left(\eta^{\otimes l}\otimes\xi^{\otimes m}\right), \qquad \xi,\eta\in E_{\mathbf{C}}, \quad \kappa\in E_{\mathbf{C}}^{\otimes(l+m)}.$$

In particular, for $\xi,\eta\in E_{\mathbf{C}}$ we have

$$(\partial_t\otimes I)\widehat{\ }(\xi,\eta) = e^{\langle\xi,\eta\rangle}\xi(t), \qquad (\partial_t^*\otimes I)\widehat{\ }(\xi,\eta) = e^{\langle\xi,\eta\rangle}\eta(t),$$

where the right hand sides are scalar operators on $\mathcal{E}$, see also §4.1.

We now list some properties of the symbol of an operator Ξ in $\mathcal{L}((E)\otimes\mathcal{E}, ((E)\otimes\mathcal{E})^*)$ and $\mathcal{L}((E)\otimes\mathcal{E}, (E)\otimes\mathcal{E})$. Put $\Theta=\widehat{\Xi}$.

(O1) (analyticity) For any $\xi,\xi_1,\eta,\eta_1\in E_{\mathbf{C}}$ and $u,v\in\mathcal{E}$ the function

$$z,w\mapsto\langle\Theta(z\xi+\xi_1, w\eta+\eta_1)u, v\rangle, \qquad z,w\in\mathbf{C},$$

is entire holomorphic;

(O2) (boundedness) There exist constant numbers $C\ge 0$, $K\ge 0$ and $p\in\mathbf{R}$ such that

$$|\langle\Theta(\xi,\eta)u, v\rangle| \le C\,|u|_p\,|v|_p \exp K\left(|\xi|_p^2+|\eta|_p^2\right), \qquad \xi,\eta\in E_{\mathbf{C}}, \quad u,v\in\mathcal{E}.$$

If $\Xi \in \mathcal{L}((E)\otimes\mathcal{E}, (E)\otimes\mathcal{E})$, we have

(O2′) (boundedness) For any $p \geq 0$ and $\epsilon > 0$ there exist $C \geq 0$ and $q \geq 0$ such that

$$|\langle \Theta(\xi,\eta)u\,,v\rangle| \leq C\,|u|_{p+q}\,|v|_{-p}\exp\epsilon\left(|\xi|_{p+q}^2 + |\eta|_{-p}^2\right), \qquad \xi,\eta\in E_{\mathbf{C}}, \quad u,v\in\mathcal{E}.$$

As in the case of scalar-valued functionals (§4), we have the following characterization theorems.

THEOREM 5.7. *Let Θ be a $\mathcal{L}(\mathcal{E},\mathcal{E}^*)$-valued function on $E_{\mathbf{C}}\times E_{\mathbf{C}}$. If Θ satisfies the conditions (O1) and (O2), there exists a unique family of kernel distributions $(\kappa_{l,m})_{l,m=0}^{\infty}$, $\kappa_{l,m}\in\mathcal{L}(E_{\mathbf{C}}^{\otimes(l+m)},\mathcal{L}(\mathcal{E},\mathcal{E}^*))_{\mathrm{sym}(l,m)}$, such that*

$$\langle \Theta(\xi,\eta)u\,,v\rangle = \sum_{l,m=0}^{\infty}\left\langle\!\left\langle \Xi_{l,m}(\kappa_{l,m})(\phi_\xi\otimes u)\,,\phi_\eta\otimes v\right\rangle\!\right\rangle, \qquad \xi,\eta\in E_{\mathbf{C}}.$$

Moreover, the series

$$\Xi\phi = \sum_{l,m=0}^{\infty}\Xi_{l,m}(\kappa_{l,m})\phi, \qquad \phi\in(E)\otimes\mathcal{E}, \tag{5-7}$$

converges in $((E)\otimes\mathcal{E})^*$, $\Xi\in\mathcal{L}((E)\otimes\mathcal{E},((E)\otimes\mathcal{E})^*)$ and $\widehat{\Xi}=\Theta$. If Θ satisfies (O1) and (O2′), the kernel distribution $\kappa_{l,m}$ belongs to $\mathcal{B}((E_{\mathbf{C}}^{\otimes l})^*, E_{\mathbf{C}}^{\otimes m};\mathcal{L}(\mathcal{E},\mathcal{E}))_{\mathrm{sym}(l,m)}$. In that case the series (5-7) converges in $(E)\otimes\mathcal{E}$ and $\Xi\in\mathcal{L}((E)\otimes\mathcal{E},(E)\otimes\mathcal{E})$.

5.7. Fock expansion. As an immediate consequence of Theorem 5.7 we come to the Fock expansion of $\Xi\in\mathcal{L}((E)\otimes\mathcal{E},((E)\otimes\mathcal{E})^*)$.

THEOREM 5.8. *For any $\Xi\in\mathcal{L}((E)\otimes\mathcal{E},((E)\otimes\mathcal{E})^*)$ there exists a unique family of distributions $(\kappa_{l,m})_{l,m=0}^{\infty}$, $\kappa_{l,m}\in\mathcal{L}(E_{\mathbf{C}}^{\otimes(l+m)},\mathcal{L}(\mathcal{E},\mathcal{E}^*))_{\mathrm{sym}(l,m)}$, such that*

$$\Xi\phi = \sum_{l,m=0}^{\infty}\Xi_{l,m}(\kappa_{l,m})\phi, \qquad \phi\in(E)\otimes\mathcal{E}, \tag{5-8}$$

where the right hand side converges in $((E)\otimes\mathcal{E})^*$. If $\Xi\in\mathcal{L}((E)\otimes\mathcal{E},(E)\otimes\mathcal{E})$, then every kernel distribution $\kappa_{l,m}$ belongs to $\mathcal{B}((E_{\mathbf{C}}^{\otimes l})^*, E_{\mathbf{C}}^{\otimes m};\mathcal{L}(\mathcal{E},\mathcal{E}))_{\mathrm{sym}(l,m)}$ and the right hand side of (5-8) converges in $(E)\otimes\mathcal{E}$.

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